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Implementation of the U(n) tensor operator calculus in a vector Bargmann Hilbert space

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Abstract. Using a vector coherent state framework and an associated vector Bargmann Hilbert space, the structure of two classes of $\mathrm{U}(n):\mathrm{U}(n-1)$ unit projective operators is shown to be intimately related to (6-j) and (9-j) coefficients of the $\mathrm{U}(n-1)$ subalgebra. Explicit verification of limit properties for these operators allows the unambiguous assignment of a set of canonical operator labels.

1. Introduction

The existence of the canonical decomposition chain

$$\mathrm{U}(n) \supset \mathrm{U}(n-1) \supset \ldots \supset \mathrm{U}(2) \supset \mathrm{U}(1)$$

for the unitary group U(n) has a very important consequence regarding the construction of unit tensor operators for U(n) or, equivalently, for the computation of coupling coefficients symmetry adapted to the canonical chain: such tensor operators can be constructed in a modular fashion in terms of a set of imbricated unit projective tensor operators for $\mathrm{U}(i+1)$: U(i) (Louck and Biedenharn 1970). These projective operators are, by definition, U(i)-invariant operators.

Consider, for specificity, the $\mathrm{U}(n):\mathrm{U}(n-1)$ unit projective operator or, equivalently, the $\mathrm{U}(n):\mathrm{U}(n-1)$ reduced Wigner coefficients. One might then ask if this projective operator, a $\mathrm{U}(n-1)$ invariant operator, can be constructed, in a simple multiplicative fashion, in terms of (6-j) and (9-j) recoupling operators for $\mathrm{U}(n-1)$, a known class of $\mathrm{U}(n-1)$ invariant operators, and some well defined $\mathrm{U}(n):\mathrm{U}(n-1)$ invariant normalisation coefficients K with simple limit properties. Such a question is motivated by the recent realisation (Le Blanc and Hecht 1987) that the elementary unit projective operators for U(n) (Biedenharn and Louck 1968) can be written down rather simply in terms of $\mathrm{U}(n-1)$ unitary Racah coefficients and K normalisation factors. This result also applies to a restricted class of $\mathrm{U}(3):\mathrm{U}(2)$ isoscalar coefficients (Le Blanc 1987). A major result of the present paper is to develop these generalisations for two large classes of unit tensor operators in all U(n).

The concept which allows the demonstration of this gratifying result is that of induction of finite-dimensional irreducible representations for U(n) from representations of its canonical subgroup $\mathrm{U}(n-1)$. This induction process is embodied in a most transparent fashion in the theory of vector coherent states (Deenen and Quesne 1984, Rowe 1984, Rowe *et al* 1985, 1988, Le Blanc and Rowe 1988a, b). Vector coherent state (VCS) theory prescribes the rules for an expansion of the Lie algebra u(n) of

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U(n) in terms of a set of linear differential operators acting on a space of vector-valued holomorphic functions. Upon introduction of an appropriate scalar product, this space is referred to as a vector Bargmann (VB) Hilbert space (Rowe 1984, Le Blanc and Rowe 1988b). Significantly, the VCS expansion belongs to the enveloping algebra of a contraction of u(n) (the contraction of its semisimple part su(n) $\to$ u($n-1$)$\oplus$hw($n-1$)); VCS theory therefore offers an optimal framework for the study of limit properties of the projective operators in the contraction limit.

The structure of this paper is follows. In § 2, we review VCS theory for Gel'fand-Weyl bases of U(n). In § 3, we extend VCS theory to U(n) unit tensor operators acting on the VB Hilbert space. In § 4, matrix elements for two general classes of tensor operators in U(n) are evaluated as closed algebraic expressions for U(n):U($n-1$) unit projective operators involving U($n-1$) unitary (9-j) coefficients and VCS normalisation coefficients K. Finally, in § 5, we study the limit properties of these results under contraction of the u(n) Lie algebra and justify the assignment of specific tensor operator patterns to these VCS tensor operators.

2. VCS construction of U(n) Gel'fand–Weyl bases

The detailed prescription for the VCS construction of G $\supset$ H symmetry-adapted bases for ladder representations of a group G and a subgroup H of G can be found in Rowe *et al* (1988) and Le Blanc and Rowe (1988a, b). This construction is a generalisation of standard (scalar) coherent state theory (Perelomov 1972, 1986, Onofri 1975). We review in this section the VCS realisation of the unitary Lie group U(n) with Lie algebra u(n) generated by the set of generators $\{E_{ij}\,;\,1\leqslant i,j\leqslant n\}$ obeying the standard u(n) commutator algebra.

Since we are interested in the Gel'fand basis of U(n), we set G = U(n) = SU(n)$\otimes$U(1)$_n$ and H = U($n-1$)$\otimes$U(1)$_n$, where the Abelian group U(1)$_n$ is generated by the nth weight operator E_{nn}. (The more general construction U($m+n$) $\supset$ U(m)$\otimes$U(n) and its supersymmetric counterpart U(m/n) $\supset$ U(m)$\otimes$U(n) have recently been given by Le Blanc and Rowe (1988b).) Consequently, we decompose the Lie algebra u(n) into four subsets:

(a) the u($n-1$) subalgebra†

$$\mathrm{u}(n-1)=\mathrm{span}\{C_{\alpha\beta}=E_{\alpha\beta}\,;\,1\leqslant\alpha,\beta\leqslant n-1\} \tag{2.1a}$$

(b) the Abelian subalgebra u(1)$_n$

$$\mathrm{u}(1)_n=\mathrm{span}\{E_{nn}\} \tag{2.1b}$$

(c) an Abelian nilpotent subalgebra of raising operators

$$\boldsymbol{n}_+=\mathrm{span}\{A_\alpha=E_{\alpha n}\,;\,1\leqslant\alpha\leqslant n-1\} \tag{2.1c}$$

(d) an Abelian nilpotent subalgebra of lowering operators

$$\boldsymbol{n}_-=\mathrm{span}\{B_\alpha=E_{n\alpha}\,;\,1\leqslant\alpha\leqslant n-1\}. \tag{2.1d}$$

Consider a linear unitary irreducible representation (unirrep) of U(n) defined by the Young frame‡ $[\boldsymbol{m}_n]\equiv[m_{1n}m_{2n}\ldots m_{nn}]$. The carrier space of $[\boldsymbol{m}_n]$ will be denoted

† Throughout this paper, we adopt the convention that latin indices run from 1 to n while greek indices run from 1 to $n-1$.

‡ More generally, we set $[\boldsymbol{m}_j]\equiv[m_{1j},m_{2j},\ldots,m_{jj}]$.

by the set of vectors $\{|(m)_n\rangle\}$, where $(m)_n$ is an n-rowed Gel'fand–Weyl pattern (Baird and Biedenharn 1963, Louck 1970). The VCS approach, which is closely related to the induction construction of Mackey (1968), focuses, for a highest weight unirrep, on the subset of vectors in $[\boldsymbol{m}_n]$ annihilated by the raising operator algebra $\boldsymbol{n}_+$, i.e. on the subset

$$\{|(u)_{n-1}\rangle\} \equiv \{|(m)_n\rangle \in [\boldsymbol{m}_n] \text{ such that } A_\alpha|(m)_n\rangle = 0, \forall A_\alpha \in \boldsymbol{n}_+\}. \tag{2.2}$$

This subset of vectors carries the irrep

$$[\boldsymbol{u}_{n-1}]\otimes[m_{nn}] = [u_{1,n-1}, u_{2,n-1} \ldots, u_{n-1,n-1}]\otimes[m_{nn}] \qquad \text{with } u_{i,n-1} = m_{in}$$

of the stability subgroup $\mathrm{H} = \mathrm{U}(n-1)\otimes\mathrm{U}(1)_n$ and is called the *intrinsic space.*

A vector coherent state realisation $R(\mathcal{O})$ of a generator $\mathcal{O}\in \mathrm{u}(n)$ acts on the space of vector-valued holomorphic functions defined by

$$|(m)_n\rangle_{\mathrm{VCS}} = \sum_{(u)_{n-1}\in[\boldsymbol{u}_{n-1}]} |(u)_{n-1}\rangle\langle(u)_{n-1}|e^{z\cdot A}|(m)_n\rangle \tag{2.3}$$

where $|(m)_n\rangle$ is any state of the unirrep $[\boldsymbol{m}_n]$ of $u(n)$, $z\cdot A = \Sigma_{\alpha=1}^{n-1} z_\alpha A_\alpha$, and $\{z_\alpha\}$ is a set of $(n-1)$ complex variables used as coordinates for the coset space $\mathrm{U}(n)/(\mathrm{U}(n-1)\otimes\mathrm{U}(1)_n)$. We identify the complex variables $\{z_\alpha\}$ with boson creation operators; the set $\{\partial_\alpha \equiv \partial/\partial z_\alpha\}$ denotes their annihilation counterparts. The set of operators $\{z_\alpha, \partial_\beta, \delta_{\alpha\beta}, 1\leqslant \alpha, \beta \leqslant n-1\}$ therefore closes upon a Heisenberg–Weyl (boson) algebra $\mathrm{hw}(n-1)$ with standard commutation relations $[\partial_\alpha, z_\beta] = \delta_{\alpha\beta}$. We thus rewrite (2.3):

$$|(m)_n\rangle_{\mathrm{VCS}} = \sum_{(u)_{n-1}} \langle(u)_{n-1}|e^{z\cdot A}|(m)_n\rangle(|0\rangle\otimes|(u)_{n-1}\rangle) \tag{2.4}$$

where $|0\rangle$ is the z-boson vacuum. The VCS realisation of the kets then becomes a vector-valued function of boson operators $\{z_\alpha\}$ on the vacuum.

The basis $\{|(m)_n\rangle_{\mathrm{VCS}}\}$ carries the $\mathrm{U}(n)$ irrep $[\boldsymbol{m}_n]$. Let g belong to $\mathrm{G} = \mathrm{U}(n)$. We define the action

$$g|(m)_n\rangle \equiv D(g)|(m)_n\rangle = \sum_{(m')_n} D^{[m_n]}_{(m')_n,(m)_n}(g)|(m')_n\rangle \tag{2.5}$$

with $g\to D(g)$ defining the (matrix) irrep $[\boldsymbol{m}_n]$ on the abstract basis $\{|(m)_n\rangle\}$. To show that the VCS basis (2.4) carries the same irrep, we use the same group action:

$$g\circ|(m)_n\rangle_{\mathrm{VCS}} \equiv \sum_{(u)_{n-1}} \langle(u)_{n-1}|\, \mathrm{e}^{z\cdot A}g|(m)_n\rangle(|0\rangle\otimes|(u)_{n-1}\rangle). \tag{2.6}$$

It follows from (2.5) that

$$\begin{aligned} g\circ|(m)_n\rangle_{\mathrm{VCS}} &\equiv \sum_{(m')_n} D^{[m_n]}_{(m')_n,(m)_n}(g) \sum_{(u)_{n-1}} \langle(u)_{n-1}|\, \mathrm{e}^{z\cdot A}|(m')_n\rangle)(|0\rangle\otimes|(u)_{n-1}\rangle) \\ &= \sum_{(m')_n} D^{[m_n]}_{(m')_n,(m)_n}(g)|(m')_n\rangle_{\mathrm{VCS}} \end{aligned} \tag{2.7}$$

which establishes the desired result.

Consider now the action of the Lie algebra $\mathrm{u}(n)$ on the basis $\{|(m)_n\rangle_{\mathrm{VCS}}\}$. Let $\mathcal{O}\in\mathrm{u}(n)$. From (2.6), we see that

$$\begin{aligned} &R(\mathcal{O})|(m)_n\rangle_{\mathrm{VCS}} \\ &\quad\equiv \mathcal{O}\circ|(m)_n\rangle_{\mathrm{VCS}} \\ &\quad= \sum_{(u)_{n-1}} \langle(u)_{n-1}|\, \mathrm{e}^{z\cdot A}\mathcal{O}|(m)_n\rangle(|0\rangle\otimes|(u)_{n-1}\rangle) \\ &\quad= \sum_{(u)_{n-1}} \langle(u)_{n-1}|\left(\mathcal{O} + \frac{1}{1!}[z\cdot A, \mathcal{O}] + \frac{1}{2!}[z\cdot A, [z\cdot A, \mathcal{O}]] + \ldots\right) \\ &\qquad\times \mathrm{e}^{z\cdot A}|(m)_n\rangle(|0\rangle\otimes|(u)_{n-1}\rangle). \end{aligned} \tag{2.8}$$

One then finds, with the usual convention concerning the summation of repeated indices, the following VCS expansion for the Lie algebra u(n) of U(n) (Hecht *et al* 1987, Le Blanc and Rowe 1988b):

$$R(A_\alpha)=\partial_\alpha\equiv\frac{\partial}{\partial z_\alpha} \tag{2.9a}$$

$$R(C_{\alpha\beta})=\mathscr{C}_{\alpha\beta}-z_\beta\partial_\alpha \tag{2.9b}$$

$$R(B_\alpha)=z_\gamma\mathscr{C}_{\gamma\alpha}-z_\alpha\mathscr{E}_{nn}-z_\alpha z_\gamma\partial_\gamma \tag{2.9c}$$

$$R(E_{nn})=\mathscr{E}_{nn}+z_\alpha\partial_\alpha \tag{2.9d}$$

where the action of the stability algebra on the intrinsic space defines the $\mathrm{u}(n-1)\oplus\mathrm{u}(1)_n$ intrinsic algebra $\mathrm{span}\{\mathscr{C}_{\alpha\beta}\}\oplus\mathrm{span}\{\mathscr{E}_{nn}\}$:

$$\langle(u)_{n-1}|\mathscr{C}_{\alpha\beta}\,\mathrm{e}^{z\cdot A}|(m)_n\rangle\equiv\langle(u)_{n-1}|C_{\alpha\beta}\,\mathrm{e}^{z\cdot A}|(m)_n\rangle \qquad C_{\alpha\beta}\in\mathrm{u}(n-1)$$

$$\langle(u)_{n-1}|\mathscr{E}_{nn}\,\mathrm{e}^{z\cdot A}|(m)_n\rangle\equiv\langle(u)_{n-1}|E_{nn}\,\mathrm{e}^{z\cdot A}|(m)_n\rangle.$$

Note that in the VCS realisation, the $\mathrm{u}(n-1)$ subalgebra (2.9b) consists of the piecewise sum by component of the intrinsic $\mathrm{u}(n-1)$ subalgebra $\mathscr{C}_{\alpha\beta}$ and a variant $(-z_\beta\partial_\alpha)$ of the Jordan-Schwinger boson realisation of $\mathrm{u}(n-1)$. The two realisations *commute*, so that the vectors in $\{|(m)_n\rangle_{\mathrm{VCS}}\}$ with sharp $\mathrm{U}(n-1)$ labels are defined (see (2.11) and (2.12) below) by $\mathrm{U}(n-1)$ vector couplings†.

The set of VCS operators $\{\partial_\alpha\}$, like $\{A_\alpha\}$, carries the $\mathrm{U}(n-1)$ irrep $[1\dot{0}]$. From (2.9b), it follows that $\{z_\alpha\}$ carries a $\mathrm{U}(n-1)$ irrep $[\dot{0}-1]$. Using this fact, we see that the direct product $[\dot{0}-1]\times\ldots\times[\dot{0}-1]$ of w such irreps defines uniquely the $\mathrm{U}(n-1)$ irrep $[\dot{0}-w]$ as symmetric polynomials in $\{z_\alpha\}$ of rank w. Let us denote the vectors of this irrep, normalised in the standard (boson) way, by

$$\left|\begin{matrix}[\dot{0}-w]\\(-\lambda)_{n-2}\end{matrix}\right\rangle=Z^{[\dot{0}-w]}_{(-\lambda)_{n-2}}(z)|0\rangle \tag{2.10}$$

where $(-\lambda)_{n-2}$ denotes the $\mathrm{U}(n-2)$ Gel'fand-Weyl pattern labelling the $\mathrm{U}(n-1)$ irrep $[\dot{0}-w]$ and where $Z^{[\dot{0}-w]}(z)$ is a fully symmetric and normalised polynomial of rank w in $\{z_\alpha\}$. The vectors in the VCS basis having prescribed $\mathrm{U}(n-1)$ labels $(m)_{n-1}$ are then, to within a phase and a normalisation coefficient (see (2.12) below), given by

$$\left[\left|\begin{matrix}[\dot{0}-w]\\(\cdot)\end{matrix}\right\rangle\times\left|\begin{matrix}[\boldsymbol{u}_{n-1}]\\(\cdot)\end{matrix}\right\rangle\right]_{(m)_{n-1}}\otimes[m_{nn}] \tag{2.11}$$

where the square bracket represents a $\mathrm{U}(n-1)$ coupling and $(m)_{n-1}$ denotes the $\mathrm{U}(n-1)$ Gel'fand-Weyl subpattern for the $\mathrm{U}(n-1)$ subirrep $[\boldsymbol{m}_{n-1}]$ belonging to the $\mathrm{U}(n)$ irrep $[\boldsymbol{m}_n]$.

Although the states (2.11) are properly normalised in $\mathrm{U}(n-1)$ with respect to the vector Bargmann (VB) measure (Bargmann 1961, Le Blanc and Rowe 1988b), this normalisation is not consistent with the action of the VCS operators R (2.9) for the standard $\mathrm{U}(n)$ unitary representation $D(g)$ defined in (2.7). According to Weyl's celebrated theorem, we know that this realisation is nevertheless equivalent to the standard unitary realisation. Upon introduction of a similarity transformation K on the VB Hilbert space (allowing us to retain the simple VB measure and still retrieve

† This observation readily explains the appearance of a (9-j) symbol in the expressions (4.3) and (4.6) for two classes of $\mathrm{U}(n):\mathrm{U}(n-1)$ unit projective operators.

unitarity (Rowe 1984, Rowe *et al* 1988, Le Blanc and Rowe 1988b)), one then obtains the following explicit form for the VCS vectors of a general U(n) irrep:

$$|(m)_n\rangle_{\mathrm{VCS}} = (-1)^{\phi} K\begin{pmatrix} [\boldsymbol{m}_n] \\ [\boldsymbol{m}_{n-1}] \end{pmatrix} [Z^{[\dot{0}-w]}_{(\cdot)}(z) \times B^{[\boldsymbol{m}_n]}_{(\cdot)}(a)]_{(m)_{n-1}}|0\rangle \tag{2.12}$$

where

(a) $Z^{[\dot{0}-w]}(z)$ are normalised boson polynomials of rank w in the $\{z_\alpha\}$ carrying the U($n-1$) irrep $[\dot{0}-w]$,

(b) the rank w, eigenvalue of the operator $W \equiv (E_{nn} - m_{nn}\cdot \mathbf{1})$, is given by

$$w = \sum_{i=1}^{n-1} (m_{i,n} - m_{i,n-1}) \tag{2.13}$$

(c) $\{B^{[\boldsymbol{m}_n]}_{(\cdot)}(a)\}$ denotes the set of intrinsic bosonic states carrying the U($n-1$)$\otimes$ U(1)$_n$ irrep

$$\begin{aligned}[][\boldsymbol{u}_{n-1}]\otimes[m_{nn}] &= [u_{1,n-1}u_{2,n-1}\ldots, u_{n-1,n-1}]\otimes[m_{nn}] \\ &= [m_{1n}m_{2n}\ldots, m_{n-1,n}]\otimes[m_{nn}].\end{aligned}$$

For definiteness, $B^{[\boldsymbol{m}_n]}(a) = B^{[\boldsymbol{u}_{n-1}]}(a^\alpha_\beta)\otimes B^{[m_{nn}]}(a_n)$ is the product of two boson polynomials with $B^{[\boldsymbol{u}_{n-1}]}(a^\alpha_\beta)$ a normalised boson polynomial (Louck and Biedenharn 1973) in the boson creation operators a^α_β, $1\leq \alpha, \beta \leq n-1$, and the monomial $B^{[m_{nn}]}(a_n)$ simply given by

$$B^{[m_{nn}]}(a_n) = \frac{a_n^{m_{nn}}}{(m_{nn}!)^{1/2}}.$$

The u($n-1$)$\oplus$u(1)$_n$ intrinsic algebra is then given by the Jordan-Schwinger realisation

$$\mathscr{C}_{\alpha\beta} = \sum_\sigma a^\sigma_\alpha \bar{a}^\sigma_\beta \qquad \mathscr{E}_{nn} = a_n\bar{a}_n \tag{2.14}$$

where $\bar{a}$ denotes a boson annihilation operator.

(d) $|0\rangle$ now stands for both the z- and a-boson vacua.

(e) The bracket [...×...], as in (2.11), denotes a U($n-1$) vector coupling.

(f) The U($n-1$)×U(1)$_n$ invariant normalisation factor K is given by (Hecht *et al* 1987)

$$K^2\begin{pmatrix} [\boldsymbol{m}_n] \\ [\boldsymbol{m}_{n-1}] \end{pmatrix} = \prod_{\alpha=1}^{n-1} \frac{(p_{\alpha n} - p_{nn} - 1)!}{(p_{\alpha,n-1} - p_{nn})!} \tag{2.15}$$

where the partial hook p_{ij} is defined by $p_{ij} = m_{ij} + j - i$. We promote K to an Hermitian operator, $K_{\mathrm{op}} = K^\dagger_{\mathrm{op}}$, with matrix elements, diagonal in U($n-1$), given by (2.15). The generators of the stability algebra U($n-1$)$\otimes$U(1)$_n$, i.e. $R(C_{\alpha\beta})$ and $R(E_{nn})$, then commute with K_{op}.

(f) The phase factor ϕ is given by

$$\phi = \phi([\boldsymbol{u}_{n-1}]) - \phi([\dot{0} - w]) - \phi([\boldsymbol{m}_{n-1}]). \tag{2.16}$$

(See Hecht *et al* (1987, appendix) for definition and properties of the phase factor ϕ. See also (4.3c).) With this phase convention, which agrees with the standard one, we shall consider only the positive square root of K^2.

The importance of the VCS construction, which is successfully exploited in the remaining sections, lies in the fact that it yields a realisation of the $u(n)$ Lie algebra and, more important, a construction of the $U(n)$ carrier space defined entirely within the context of $U(n-1)$ vector addition coefficients. In other words, the VCS construction allows one to obtain $U(n)$ results with $U(n-1)$ methods, and is accordingly a most useful technique in a recursive approach to the set of all $U(n)$ groups.

Finally, if one defines the (equivalent) realisation

$$\rho(\mathscr{C}) = K^{-1}R(\mathscr{C})K \tag{2.17a}$$

for $u(n)$, the corresponding representation is then carried by the VB vectors

$$\{|(m)_n\rangle_{\mathrm{VB}} = K^{-1}|(m)_n\rangle_{\mathrm{VCS}}\} \tag{2.17b}$$

which are explicitly orthonormal under the VB measure (this approach has been consistently advocated by Rowe). The representation ρ is then unitary under the usual boson conjugation rules $(z)^\dagger = \partial/\partial z$, $(a^\dagger)^\dagger = \bar{a}$. Hermiticity properties of the various operators introduced below will not be discussed explicitly. Nevertheless, they can easily be derived using the ρ realisation for $u(n)$ and would amount to a re-derivation of some well known symmetry properties of (multiplicity-free) coupling coefficients for $U(n)$ (see, e.g., Hecht *et al* 1987, appendix.)

3. U(*n*) tensor operators in a VCS framework

The construction of a canonical basis for tensor operators in $U(n)$ relies heavily on two properties: the *equivariance property* and the *derivation property*.

(*a*) The equivariance property. Let t denote a tensor operator in $U(n)$. Then t is a *set* of operators $t = \{t(m)_n\}$ obeying the defining relation for equivariance,

$$g : t(m)_n \to t'(m)_n \equiv g(t(m)_n)g^{-1} = \sum_{(m')_n} D^{[m_n]}_{(m')_n,(m)_n}(g)t(m')_n \tag{3.1}$$

for $g \in U(n)$ and $(m)_n$, $(m')_n$ Gel'fand-Weyl patterns in $U(n)$, labelling a basis for the irrep $[\boldsymbol{m}_n]$. Unit tensor operators, which form an operator basis for all tensor operators, can be canonically realised by matrices in the space $\mathscr{H} \times \mathscr{H}^{\text{dual}}$ where $\mathscr{H}$ is the Hilbert space $\mathscr{H} = \Sigma_{[\boldsymbol{m}_n]} \oplus \mathscr{H}([\boldsymbol{m}_n])$, the direct sum of carrier spaces for each unirrep $[\boldsymbol{m}_n]$ of $U(n)$ taken once and only once, subject to the equivalence constraint $D^{[m_{1n}+m_{nn}, m_{2n}+m_{nn}, \ldots, m_{nn}]} \simeq D^{[m_{1n}, m_{2n}, \ldots, 0]}$. This construction for $\mathscr{H}$ has been termed a 'model space' for $U(n)$ by Gel'fand and Zelevinsky (1985).

(*b*) The derivation property. Let $\mathscr{C}$ belong to the Lie algebra $u(n)$. A realisation of $u(n)$ by linear operators has the derivation property if and only if

$$\mathscr{C}(AB) = (\mathscr{C}A)B + A(\mathscr{C}B) \tag{3.2}$$

where AB denotes the tensor product of two representations of $U(n)$, including (using equivariance) tensor operators.

Unit tensor operators can be given a canonical matrix realisation as follows. Define a unit tensor operator t by the symbol

$$\left\langle \begin{matrix} (\Gamma)_{n-1} \\ [\boldsymbol{m}_n^{(t)}] \\ (m^{(t)})_{n-1} \end{matrix} \right\rangle$$

where

$$\begin{pmatrix} [\boldsymbol{m}_n^{(t)}] \\ (m^{(t)})_{n-1} \end{pmatrix}$$

is the Gel'fand–Weyl pattern associated with the vector $(m^{(t)})_n$ of the irrep $[\boldsymbol{m}_n^{(t)}]$, and $(\Gamma)_{n-1}$ is an operator (inverted) pattern which canonically resolves all multiplicity (Louck and Biedenharn 1970). Such a unit tensor operator t in $\mathrm{U}(n)$ has an action on a generic state vector that effects a change in the $\mathrm{U}(n)$ representation labels, i.e. $t:[\boldsymbol{m}_n]\to[\boldsymbol{m}_n]+[\Delta(\Gamma)]$, where $[\Delta(\Gamma)]=[\Delta_1(\Gamma),\Delta_2(\Gamma),\ldots,\Delta_n(\Gamma)]$ denotes the label shifts $\Delta_i(\Gamma)$ (Biedenharn and Louck 1968, Louck 1970) and the sum is done componentwise ($m_{in}\to m_{in}+\Delta_i(\Gamma)$). The matrices of the unit tensor operator $\langle[\boldsymbol{m}_n^{(t)}]\rangle$ are then the $\mathrm{U}(n)$ vector addition coefficients:

$$\left\langle \begin{matrix} [\boldsymbol{m}_n^{(f)}] \\ (\boldsymbol{m}^{(f)})_{n-1} \end{matrix} \middle| \left\langle \begin{matrix} (\Gamma)_{n-1} \\ [\boldsymbol{m}_n^{(t)}] \\ (m^{(t)})_{n-1} \end{matrix} \right\rangle \middle| \begin{matrix} [\boldsymbol{m}_n^{(i)}] \\ (\boldsymbol{m}^{(i)})_{n-1} \end{matrix} \right\rangle$$

where $[\boldsymbol{m}^{(f)}]=[\boldsymbol{m}^{(i)}]+[\Delta(\Gamma)]$. (More details of this standard construction can be found in Louck and Biedenharn (1970).)

It is easily verified that the VCS realisation R (2.9) of $\mathrm{u}(n)$ has the derivation property on a product of VCS states. This indicates that the matrix realisation of unit tensor operators for $\mathrm{U}(n)$ sketched above can indeed be implemented in the VCS framework.

Consider a generic VCS state (2.12). The intrinsic space $\{|(u)_{n-1}\rangle\}$, realised in terms of a-boson polynomials $B(a)$, defines $\mathrm{U}(n-1)\otimes\mathrm{U}(1)_n$ (unnormalised) tensor operators. Since it is the transformation (equivariance) properties which are essential, we may as well replace the a-boson polynomials by normalised unit tensor operators in $\mathrm{U}(n-1)\otimes\mathrm{U}(1)_n$. Such tensor operators are canonically denoted by

$$\left\langle \begin{matrix} (\Gamma)_{n-2} \\ [\mu_{n-1}^{(t)}] \\ (\mu^{(t)})_{n-2} \end{matrix} \right\rangle \otimes \langle m_{nn}^{(t)}\rangle$$

where, for now, we require all labels belonging to the $\mathrm{U}(n-1)$ Young frame

$$[\boldsymbol{u}_{n-1}^{(t)}]=[u_{1,n-1}^{(t)}, u_{2,n-1}^{(t)}, \ldots, u_{n-1,n-1}^{(t)}]=[m_{1,n}^{(t)}, m_{2,n}^{(t)}, \ldots, m_{n-1,n}^{(t)}]$$

to be greater or equal to $m_{nn}^{(t)}$. We know, by construction, that VCS theory will promote the labels $\{u_{i,n-1}^{(t)}\}$ and $m_{nn}^{(t)}$ to the $\mathrm{U}(n)$ frame

$$[\boldsymbol{m}_n^{(t)}]=[m_{1,n}^{(t)}, m_{2,n}^{(t)}, \ldots, m_{n-1,n}^{(t)}, m_{n,n}^{(t)}].$$

The $\mathrm{U}(1)_n$ unit tensor operator $\langle m_{nn}^{(t)}\rangle$ has a trivial action on a $\mathrm{U}(1)_n$ (one-dimensional) irrep $|m_{nn}\rangle$:

$$\langle m_{nn}^{(t)}\rangle|m_{nn}\rangle=|m_{nn}+m_{nn}^{(t)}\rangle.$$

The associated unit tensor operator in VCS theory, modelled on the vector coherent state (2.12), then has the form

$$\left\langle \begin{matrix} (\Gamma)_{n-1} \\ [\boldsymbol{m}_n^{(t)}] \\ (m^{(t)})_{n-1} \end{matrix} \right\rangle_{\mathrm{VCS}} = (-1)^{\phi} K \begin{pmatrix} [\boldsymbol{m}_n^{(t)}] \\ [\boldsymbol{m}_{n-1}^{(t)}] \end{pmatrix}$$
$$\times \left(\left[Z_{(\cdot)}^{[\dot{0}-w^{(t)}]}(z) \times \left\langle \begin{matrix} (\Gamma)_{n-2} \\ [\boldsymbol{u}_{n-1}^{(t)}] \\ (\cdot)_{n-2} \end{matrix} \right\rangle \right]_{(m^{(t)})_{n-1}} \otimes \langle m_{nn}^{(t)}\rangle \right) \tag{3.3}$$

(to within an overall U(n) invariant phase) where the square bracket $[\ldots\times\ldots]_{(m^{(i)})_{n-1}}$ once more denotes a U($n-1$) coupling to the U($n-1$) Gel'fand–Weyl pattern $(m^{(t)})_{n-1}$. By construction, these operators transform equivariantly as an unirrep $[\boldsymbol{m}_n^{(t)}]$ of U(n).

The U($n-1$) operator pattern $(\Gamma)_{n-2}$ associated with the intrinsic unit operator in (3.3) induces shifts $\Delta_i(\Gamma)\geqslant\Delta_n(\Gamma)=m_{nn}^{(t)}$, $1\leqslant i\leqslant n-1$. Furthermore, the U(1)$_n$ unit tensor $\langle m_{nn}^{(t)}\rangle$ forces the nth shift $\Delta_n(\Gamma)$ to assume its *minimal* U(n) value $\Delta_n(\Gamma)=m_{nn}^{(t)}$. One must therefore assign the U(n) operator pattern

$$(\Gamma)_{n-1}=\begin{pmatrix}(\Gamma)_{n-2}\\ [\boldsymbol{u}_{n-1}^{(t)}]\end{pmatrix}=\begin{pmatrix} & & & \Gamma_{11} & & & \\ & & \cdot^{\cdot^{\cdot}} & & \ddots & & \\ & \Gamma_{1,n-2} & & \cdots & & \Gamma_{n-1,n-2} & \\ u_{1,n-1}^{(t)} & & u_{2,n-1}^{(t)} & & \cdots & & u_{n-1,n-1}^{(t)}\end{pmatrix} \tag{3.4a}$$

to the U(n) unit tensor operator in the left-hand side of (3.3). In other words, the VCS construction requires the $\Gamma_{i,n-1}$ row to be *maximally* tied to the U(n) partition $[\boldsymbol{m}_n^{(t)}]$:

$$\Gamma_{i,n-1}=u_{i,n-1}^{(t)}=m_{i,n}^{(t)}. \tag{3.4b}$$

In tableau parlance, the U(n) Young frame $[\boldsymbol{m}_n^{(t)}]$ representing the tensor, modulo the fully antisymmetrised U(n) frame

$$\underbrace{[m_{nn}^{(t)}, m_{nn}^{(t)},\ldots, m_{nn}^{(t)}]}_{n\ \text{times}}$$

distributes itself only on the first $(n-1)$ rows of the Young frame representing the initial U(n) unirrep $[\boldsymbol{m}^{(i)}]$ in the coupling

$$\left\langle\begin{matrix}(\Gamma)_{n-1}\\ [\boldsymbol{m}_n^{(t)}]\\ (\cdot)_{n-1}\end{matrix}\right\rangle_{\mathrm{VCS}}\times\left|\begin{matrix}[\boldsymbol{m}_n^{(i)}]\\ (\cdot)_{n-1}\end{matrix}\right\rangle_{\mathrm{VCS}}\rightarrow\left|\begin{matrix}[\boldsymbol{m}_n^{(f)}]\\ (\cdot)_{n-1}\end{matrix}\right\rangle_{\mathrm{VCS}}.$$

We thus have the restrictions

$$\begin{aligned}\sum_{k=1}^{n-1} m_{kn}^{(f)}&=\sum_{k=1}^{n-1} m_{kn}^{(i)}+\sum_{k=1}^{n-1}\Delta_k(\Gamma)\\ &=\sum_{k=1}^{n-1} m_{kn}^{(i)}+\sum_{k=1}^{n-1}\Gamma_{k,n-1}\\ &=\sum_{k=1}^{n-1} m_{kn}^{(i)}+\sum_{k=1}^{n-1} m_{kn}^{(t)}\end{aligned} \tag{3.5a}$$

$$m_{nn}^{(f)}=m_{nn}^{(i)}+\Delta_n(\Gamma)=m_{nn}^{(i)}+m_{nn}^{(t)}. \tag{3.5b}$$

We shall give a formal proof of the operator pattern assignment (3.4) in § 5.

The VCS tensors (3.3)—which were built in analogy to the generic VCS vectors (2.12)—are not the most general since we have used only polynomials in $\{z_\alpha\}$ and not more general polynomials over $\{z_\alpha\}$ and $\{\partial_\alpha\}$. The use of the latter allows us to ease the restriction $\Delta_n(\Gamma)=m^{(t)}$, as we now discuss.

Consider the operator ∂_α. As one of the generators of u(n), namely $R(A_\alpha)$, it is indeed a tensor operator, but it belongs to the U(n) irrep $[1\dot{0}-1]$ and not $[1\dot{0}]$ as one might like. To construct the fundamental VCS tensor operator transforming as the U(n) irrep $[1\dot{0}]$ with U(n) shift $\Delta_i(\Gamma)=\delta_{in}$, one must use (to within phase factors) the VCS operators

$$D_{\mathrm{op}}^{1/2}K_{\mathrm{op}}^{2}(\partial_\alpha\otimes\langle 1_n\rangle)K_{\mathrm{op}}^{-2}D_{\mathrm{op}}^{-1/2}\qquad \alpha=1,\ldots,n-1 \tag{3.6a}$$

and

$$D_{\mathrm{op}}^{1/2}K_{\mathrm{op}}^{2}\langle 1_n\rangle K_{\mathrm{op}}^{-2}D_{\mathrm{op}}^{-1/2} \tag{3.6b}$$

for the nth state of the $\mathrm{U}(n)$ unirrep $[1\dot{0}]$, where $\langle 1_n\rangle$ is the $\mathrm{U}(1)_n$ unit tensor, and where D_{op} is the $\mathrm{U}(n)$ dimension operator with eigenvalues

$$D([\boldsymbol{m}_n])=\frac{\Pi_{1\le r<s\le n}(p_{rn}-p_{sn})}{1!2!\ldots(n-1)!}.$$

The construction (3.6) generalises to a class of $\mathrm{U}(n)$ unit tensor operators in VCS theory. These operators are (to within phase factors) given by the expression

$$\left\langle \begin{matrix}(\Gamma)_{n-1}\\ [\boldsymbol{m}_n^{(t)}]\\ (m^{(t)})_{n-1}\end{matrix}\right\rangle_{\mathrm{VCS}} = K\begin{pmatrix}[-\boldsymbol{m}_n^{(t)}]\\ [-\boldsymbol{m}_{n-1}^{(t)}]\end{pmatrix}D_{\mathrm{op}}^{1/2}K_{\mathrm{op}}^{2}$$

$$\times\left(\left[Z_{(\cdot)}^{[v^{(t)}\dot{0}]}(\partial)\times\left\langle\begin{matrix}(\Gamma)_{n-2}\\ [m_{2n}^{(t)},m_{3n}^{(t)},\ldots,m_{nn}^{(t)}]\\ (\cdot)\end{matrix}\right\rangle\right]_{(m^{(t)})_{n-1}}\otimes\langle m_{1n}^{(t)}\rangle\right)K_{\mathrm{op}}^{-2}D_{\mathrm{op}}^{-1/2} \tag{3.7a}$$

where

$$v^{(t)}=\sum_{i=1}^{n-1}(m_{i,n-1}^{(t)}-m_{i+1,n}^{(t)}) \tag{3.7b}$$

and have been obtained by considering the Hermitian conjugate of the unit tensors (3.3) labelled by $[-\boldsymbol{m}_n^{(t)}]$, where the $\mathrm{U}(j)$ Young frame $[-\boldsymbol{m}_j]$ conjugate to $[\boldsymbol{m}_j]$ is defined by

$$[-\boldsymbol{m}_j]=[-m_{j,j},-m_{j-1,j},\ldots,-m_{1,j}].$$

Equations (3.6) and (3.7) are most easily obtained using the ρ realisation of $\mathrm{u}(n)$ defined by (2.17) (see also (4.7)).

We assign the operator pattern

$$(\Gamma)_{n-1}=\begin{pmatrix} & & & \Gamma_{11} & & & \\ & & \cdot^{\cdot^{\cdot}} & & \ddots & & \\ & \Gamma_{1,n-2} & & \cdots & & \Gamma_{n-1,n-2} & \\ m_{2n}^{(t)} & & m_{3n}^{(t)} & & \cdots & & m_{nn}^{(t)}\end{pmatrix} \tag{3.8}$$

to the operator (3.7), i.e. the $\Gamma_{i,n-1}$ row is now *minimally* tied to the $\mathrm{U}(n)$ partition $[\boldsymbol{m}_n^{(t)}]$; $\Gamma_{i,n-1}=m_{i+1,n}^{(t)}$. This implies that the nth shift assumes its *maximal* value $\Delta_n(\Gamma)=m_{1n}^{(t)}$.

As we shall show explicitly in §§ 4 and 5, the two classes (3.3) and (3.7) of tensor operators are completely defined, to within a *multiplicative* ratio of K factors which are $\mathrm{U}(n)$: $\mathrm{U}(n-1)$ invariant quantities, in terms of $\mathrm{U}(n-1)$ constructs. This is a quite remarkable result since it was entirely unexepected that any such general property could possibly hold true. These two classes of tensor operators do not, unfortunately, exhaust the set of all unit tensor operators. In this respect the construction of VCS unit tensor operators is not yet as definitive as the VCS construction of representations.

4. Evaluation of the projective operators with maximal or minimal $\Delta_n(\Gamma)$ shift

We compute in this section matrix elements for the VCS $U(n):U(n-1)$ unit projective tensor operators constructed in the previous section. Since the U(2) tensor calculus presents no inherent difficulty or multiplicity, it is relevant to first consider the case of $U(3):U(2)$ unit projective operators. We show in § 4.1 that matrix elements of these U(3) operators can be given in terms of unitary (6-j) and (9-j) recoupling coefficients for U(2). The extension of these results to the general case of the $U(n):U(n-1)$ unit tensor operator is straightforward and effected in § 4.2.

4.1. *The U(3):U(2) case*

Let us begin by recalling the definition of a projective operator (Louck and Biedenharn 1970). The concept of a projective operator has its origin in the elementary observation that a tensor operator in $U(n)$ is, at the same time, a tensor operator in the subgroup $U(n-1)$. Under the recursion hypothesis—by which one recursively generates $U(n)$ objects using knowledge of all relevant structure in $U(n-1)$—one assumes that all $U(n-1)$ unit tensor operators are known. Since the unit tensor operators in $U(n-1)$ are a basis for all tensor operators in $U(n-1)$, we may expand the $U(n)$ unit tensor operator

$$\left\langle \begin{matrix} (\Gamma)_{n-1} \\ [\boldsymbol{m}_n^{(t)}] \\ (m^{(t)})_{n-1} \end{matrix} \right\rangle = \sum_{(\gamma)_{n-2}} \left[\begin{matrix} (\Gamma)_{n-1} \\ [\boldsymbol{m}_n^{(t)}] \\ (\gamma)_{n-1} \end{matrix} \right] \left\langle \begin{matrix} (\gamma)_{n-2} \\ [\boldsymbol{m}_{n-1}^{(t)}] \\ (m^{(t)})_{n-2} \end{matrix} \right\rangle \qquad \gamma_{i,n-1} = m_{i,n-1}^{(t)} \tag{4.1}$$

where

(*a*) the object in the square brackets on the right-hand side denotes a unit projective operator on the $U(n):U(n-1)$ space (thus a $U(n-1)$ invariant operator) and

(*b*) $\langle[\boldsymbol{m}_{n-1}^{(t)}]\rangle$ is a $U(n-1)$ unit tensor operator with upper operator pattern $(\gamma)_{n-2}$. For clarity, let us note that the left-hand side of (4.1) operates on $U(n)$ vectors $|(m)_n\rangle$ where $(m)_n=(m_{ij})$, $1\leq i,j\leq n$ is an arbitrary Gel'fand-Weyl $U(n)$ pattern, whilst the right-hand side has $\langle[\boldsymbol{m}_{n-1}^{(t)}]\rangle$ acting on vectors $|(m)_{n-1}\rangle$ of the $U(n-1)$ subgroup with $(m)_{n-1}=(m_{ij})$, $1\leq i,j\leq n-1$; that is, the projective operator acts on the factor space $U(n):U(n-1)$. To be fully explicit, let us state that the matrix elements of the projective operators take the form

$$\left\langle \begin{matrix} [\boldsymbol{m}_n+\Delta(\Gamma)] \\ [\boldsymbol{m}_{n-1}+\Delta(\gamma)] \end{matrix} \right| \left[\begin{matrix} (\Gamma)_{n-1} \\ [\boldsymbol{m}_n^{(t)}] \\ (\gamma)_{n-1} \end{matrix} \right] \left| \begin{matrix} [\boldsymbol{m}_n] \\ [\boldsymbol{m}_{n-1}] \end{matrix} \right\rangle \tag{4.2}$$

where the $U(n-1)$ shifts $\Delta_i(\gamma)$ are defined similarly to the $U(n-1)$ shift $\Delta_i(\Gamma)$ shifts.

It is not difficult to evaluate the VCS unit tensor operators of § 3 when acting on the VCS vectors of § 2. Consider first the class of VCS operators (3.3) characterised by a minimal $\Delta_n(\Gamma)=m_{nn}^{(t)}$ shift. For clarity, first consider the case $n=3$. The VCS construction allows us to restrict operator manipulations entirely to U(2) couplings† and the most general case becomes a recoupling of four U(2) irreps, yielding the U(2)

† It is convenient to use here the U(2) notation since there exists a sum-of-quanta rule for the eigenvalues w of the weight operator $(E_{33}-m_{33}\cdot 1)$ made readily apparent by the VCS construction. As discussed by Biedenharn and Louck (1981b), all U(2) recoupling coefficients are identical to SU(2) recoupling coefficients: being shift invariant, one can always replace the U(2) labels $[a, b]$ by equivalent SU(2) labels $2j=(a-b)$.

version of a (9-j) unitary coefficient†. We therefore obtain for the first class of U(3) : U(2) projective operators (having $\Delta_3(\Gamma) = m_{33}^{(t)}$) the matrix element

$$
\left\langle \begin{array}{ccccc} m_{13}^{(f)} & & m_{23}^{(f)} & & m_{33}^{(f)} \\ & m_{12}^{(f)} & & m_{22}^{(f)} & \end{array} \right\| \left[\begin{array}{ccccccc} & & & \Gamma_{11} & & & \\ & & m_{13}^{(t)} & & m_{23}^{(t)} & & \\ m_{13}^{(t)} & & & m_{23}^{(t)} & & & m_{33}^{(t)} \\ & & m_{12}^{(t)} & & m_{22}^{(t)} & & \\ & & & \gamma_{11} & & & \end{array} \right]
$$

$$
\times \left| \begin{array}{ccccc} m_{13}^{(i)} & & m_{23}^{(i)} & & m_{33}^{(i)} \\ & m_{12}^{(i)} & & m_{22}^{(i)} & \end{array} \right\rangle
$$

$$
= (-1)^{(\phi([\boldsymbol{m}_3^{(i)}]) + \phi([\boldsymbol{m}_3^{(t)}]) - \phi([\boldsymbol{m}_3^{(f)}])) - (\phi([\boldsymbol{m}_2^{(i)}]) + \phi([\boldsymbol{m}_2^{(t)}]) - \phi([\boldsymbol{m}_2^{(f)}]))}
$$

$$
\times \begin{bmatrix} [m_{13}^{(i)} m_{23}^{(i)}] & [0 - w^{(i)}] & [m_{12}^{(i)} m_{22}^{(i)}] \\ [m_{13}^{(t)} m_{23}^{(t)}] & [0 - w^{(t)}] & [m_{12}^{(t)} m_{22}^{(t)}] \\ [m_{13}^{(f)} m_{23}^{(f)}] & [0 - w^{(f)}] & [m_{12}^{(f)} m_{22}^{(f)}] \end{bmatrix} \times \left[\frac{w^{(f)}!}{w^{(i)}!\, w^{(t)}!} \right]^{1/2}
$$

$$
\times K \begin{pmatrix} [\boldsymbol{m}_3^{(i)}] \\ [\boldsymbol{m}_2^{(i)}] \end{pmatrix} K \begin{pmatrix} [\boldsymbol{m}_3^{(t)}] \\ [\boldsymbol{m}_2^{(t)}] \end{pmatrix} \left[K \begin{pmatrix} [\boldsymbol{m}_3^{(f)}] \\ [\boldsymbol{m}_2^{(f)}] \end{pmatrix} \right]^{-1} \tag{4.3a}
$$

where $\Gamma_{11} = m_{13}^{(f)} - m_{13}^{(i)}$, $\gamma_{11} = m_{12}^{(f)} - m_{12}^{(i)}$ and

$$
w^{(f)} = w^{(i)} + w^{(t)} \tag{4.3b}
$$

$$
\phi([\boldsymbol{m}_j]) = \tfrac{1}{2} \sum_{1 \leqslant k \leqslant l \leqslant j} (m_{kj} - m_{lj}) = \tfrac{1}{2} \sum_{1 \leqslant k \leqslant j} (j + 1 - 2k) m_{kj}. \tag{4.3c}
$$

Thus the most general U(3) : U(2) reduced Wigner coefficient obeying the constraint‡ (3.5) is easily written down in terms of three simple constitutive blocks: a (9-j) recoupling coefficient for U(2) (Jucys and Bandzaitis 1977, Biedenharn and Louck 1981a), a ratio of w factors and a ratio of K factors.

It is useful to remark that shift invariance ($m_{ij} \simeq m_{ij} + \text{constant}$) applies independently to each of the three constitutive blocks in the matrix element on the right-hand side of (4.3). Each of the U(2) patterns $[a, b]$ in the (9-j) coefficient on the right-hand side represents an angular momentum $(a-b)/2$ and, as such, is shift invariant. Similarly, the equalities defining w and K are explicitly shift invariant. This observation is just a restatement of the equivalence relation (3.2) applied to (4.3).

An interesting special case of this general result occurs when $m_{12}^{(t)} = m_{13}^{(t)}$, $m_{22}^{(t)} = m_{23}^{(t)}$ implying $w^{(i)} = w^{(f)} = w$, $w^{(t)} = 0$. Equation (4.3) then simplifies to

$$
\left\langle \begin{array}{ccccc} m_{13}^{(f)} & & m_{23}^{(f)} & & m_{33}^{(f)} \\ & m_{12}^{(f)} & & m_{22}^{(f)} & \end{array} \right\| \left[\begin{array}{ccccccc} & & & \Gamma_{11} & & & \\ & & m_{13}^{(t)} & & m_{23}^{(t)} & & \\ m_{13}^{(t)} & & & m_{23}^{(t)} & & & m_{33}^{(t)} \\ & & m_{13}^{(t)} & & m_{23}^{(t)} & & \\ & & & \gamma_{11} & & & \end{array} \right]
$$

$$
\times \left| \begin{array}{ccccc} m_{13}^{(i)} & & m_{23}^{(i)} & & m_{33}^{(i)} \\ & m_{12}^{(i)} & & m_{22}^{(i)} & \end{array} \right\rangle
$$

$$
= (-1)^{(\phi([\boldsymbol{m}_3^{(i)}]) + \phi([\boldsymbol{m}_3^{(t)}]) - \phi([\boldsymbol{m}_3^{(f)}])) - (\phi([\boldsymbol{m}_2^{(i)}]) + \phi([\boldsymbol{u}_2^{(t)}]) - \phi([\boldsymbol{m}_2^{(f)}]))}
$$

$$
\times \begin{bmatrix} [m_{13}^{(i)} m_{23}^{(i)}] & [0 - w] & [m_{12}^{(i)} m_{22}^{(i)}] \\ [m_{13}^{(t)} m_{23}^{(t)}] & [0] & [m_{13}^{(t)} m_{23}^{(t)}] \\ [m_{13}^{(f)} m_{23}^{(f)}] & [0 - w] & [m_{12}^{(f)} m_{22}^{(f)}] \end{bmatrix} \times K \begin{pmatrix} [\boldsymbol{m}_3^{(i)}] \\ [\boldsymbol{m}_2^{(i)}] \end{pmatrix} \left[K \begin{pmatrix} [\boldsymbol{m}_3^{(f)}] \\ [\boldsymbol{m}_2^{(f)}] \end{pmatrix} \right]^{-1} \tag{4.4a}
$$

† This result was anticipated. See remarks following (2.9).

‡ Note that the definition (2.13) for w and the constraint (3.5) imply (4.3b).

$$
\begin{aligned}
&= (-1)^{(\phi([m_3^{(i)}])+\phi([m_3^{(t)}])-\phi([m_3^{(f)}]))-(\phi([u_2^{(i)}])+\phi([u_2^{(t)}])-\phi([u_2^{(f)}]))} \\
&\quad \times U([0-w][m_{13}^{(i)}m_{23}^{(i)}][m_{12}^{(f)}m_{22}^{(f)}][m_{13}^{(t)}m_{23}^{(t)}];\ [m_{12}^{(i)}m_{22}^{(i)}][m_{13}^{(f)}m_{23}^{(f)}]) \\
&\quad \times K\begin{pmatrix}[\boldsymbol{m}_3^{(i)}]\\ [\boldsymbol{m}_2^{(i)}]\end{pmatrix}\left[K\begin{pmatrix}[\boldsymbol{m}_3^{(f)}]\\ [\boldsymbol{m}_2^{(f)}]\end{pmatrix}\right]^{-1}
\end{aligned} \tag{4.4b}
$$

a result first derived by Le Blanc (1987). To obtain (4.4b) from (4.4a), we have used the relation

$$
\begin{aligned}
\begin{bmatrix} j_2 & j_1 & j_{12} \\ j_3 & 0 & j_3 \\ j_{23} & j_1 & j \end{bmatrix} &= U(j_1j_2jj_3;j_{12}j_{23}) \\
&= [(2j_{12}+1)(2j_{23}+1)]^{1/2}W(j_1j_2jj_3;j_{12}j_{23}).
\end{aligned} \tag{4.5}
$$

Turning our attention now to the projective operators (3.7) having maximal shift, $\Delta_3(\Gamma)=m_{13}^{(t)}$, we find

$$
\begin{aligned}
&\left\langle \begin{matrix} m_{13}^{(f)} & & m_{23}^{(f)} & & m_{33}^{(f)} \\ & m_{12}^{(f)} & & m_{22}^{(f)} & \end{matrix} \right\| \begin{bmatrix} & m_{23}^{(t)} & \Gamma_{11} & m_{33}^{(t)} & \\ m_{13}^{(t)} & & m_{23}^{(t)} & & m_{33}^{(t)} \\ & m_{12}^{(t)} & & m_{22}^{(t)} & \\ & & \gamma_{11} & & \end{bmatrix} \\
&\quad \times \left| \begin{matrix} m_{13}^{(i)} & & m_{23}^{(i)} & & m_{33}^{(i)} \\ & m_{12}^{(i)} & & m_{22}^{(i)} & \end{matrix} \right\rangle \\
&= (-1)^{m_{12}^{(t)}-m_{23}^{(t)}}\left[\frac{\dim[\boldsymbol{m}_3^{(f)}]}{\dim[\boldsymbol{m}_3^{(i)}]}\frac{\dim[\boldsymbol{m}_2^{(i)}]}{\dim[\boldsymbol{m}_2^{(f)}]}\right]^{1/2} \\
&\quad \times \begin{bmatrix} [m_{13}^{(f)}m_{23}^{(f)}] & [0-w^{(f)}] & [m_{12}^{(f)}m_{22}^{(f)}] \\ [-m_{33}^{(t)},-m_{23}^{(t)}] & [0-v^{(t)}] & [-m_{22}^{(t)},-m_{12}^{(t)}] \\ [m_{13}^{(i)}m_{23}^{(i)}] & [0-w^{(i)}] & [m_{12}^{(i)}m_{22}^{(i)}] \end{bmatrix}\left[\frac{w^{(i)}!}{w^{(f)}!\,v^{(t)}!}\right]^{1/2} \\
&\quad \times K\begin{pmatrix}[\boldsymbol{m}_3^{(f)}]\\ [\boldsymbol{m}_2^{(f)}]\end{pmatrix}K\begin{pmatrix}[-m_{33}^{(t)},-m_{23}^{(t)}]\\ [-m_{22}^{(t)},-m_{12}^{(t)}]\end{pmatrix}\left[K\begin{pmatrix}[\boldsymbol{m}_3^{(i)}]\\ [\boldsymbol{m}_2^{(i)}]\end{pmatrix}\right]^{-1}
\end{aligned} \tag{4.6a}
$$

where

$$
v^{(t)} = m_{12}^{(t)} + m_{22}^{(t)} - m_{23}^{(t)} - m_{33}^{(t)} \tag{4.6b}
$$

$$
w^{(i)} = w^{(f)} + v^{(t)}. \tag{4.6c}
$$

(This result is most easily obtained by considering matrix elements of the Hermitian conjugate tensor

$$
\left\langle \begin{matrix} & -m_{33}^{(t)} & -\Gamma_{11} & -m_{23}^{(t)} & \\ -m_{33}^{(t)} & & -m_{23}^{(t)} & & -m_{13}^{(t)} \\ & -m_{22}^{(t)} & & -m_{12}^{(t)} & \\ & & -m_{11}^{(t)} & & \end{matrix} \right\rangle \tag{4.7}
$$

which belongs to the class (3.3) of unit tensor operators.)

Of special interest is the case $m_{23}^{(t)}=m_{33}^{(t)}=0$ which corresponds to the symmetric U(3) tensor $[m_{13}^{(t)}00]$ with U(3) shifts $\Delta(\Gamma)=(00m_{13}^{(t)})$. This special case includes the elementary U(3) tensor [100] with U(3) shift (001). Due to the null entry $[-m_{33}^{(t)},-m_{23}^{(t)}]=[00]$, the (9-$j$) coefficient then collapses to a (6-j) coefficient.

4.2. The $U(n):U(n-1)$ case

Our method of deriving (4.3) and (4.6) shows that the results are not restricted to U(3), but are in fact valid for $U(n)$, *mutatis mutandis.* These results are, however, formal for $(n-1)>3$ since the necessary canonical definitions for the $U(n-1)$ recoupling coefficients and, more generally, the $U(n-1)$ tensor operator structures have been given in complete form for U(3) only. Since we are confident that this lacuna can be remedied, and since the form of the results would be maintained, it is then of some value to be explicit about this $U(n)$ result, as we now discuss.

Clearly, the left-hand sides of (4.3a) and (4.6a) generalise at once to $U(n)$, as do the K factors and the various parameters w. The only technical point in this generalisation concerns the $U(n-1)$ analogues to the (6-j) and (9-j) recoupling coefficients. For the general case, the (6-j) symbol must be augmented by four operator patterns, and the (9-j) by six operator patterns (see also § 5). In other words, each independent coupling defining the (n-j) coefficient must be supplemented by the proper operator pattern. Fortunately, the three rows and the middle column in the (9-j) of both (4.3) and (4.6) involve only multiplicity-free couplings and the corresponding operator patterns are uniquely determined by the shifts and thus redundant. The coupling represented by columns 1 and 3, however, do require operator patterns. We thus write

$$\begin{bmatrix} [\boldsymbol{u}^{(i)}_{n-1}] & [\dot{0}-w^{(i)}] & [\boldsymbol{m}^{(i)}_{n-1}] \\ [\boldsymbol{u}^{(t)}_{n-1}] & [\dot{0}-w^{(t)}] & [\boldsymbol{m}^{(t)}_{n-1}] \\ (\Gamma)_{n-2} & & (\gamma)_{n-2} \\ [\boldsymbol{u}^{(f)}_{n-1}] & [\dot{0}-w^{(f)}] & [\boldsymbol{m}^{(f)}_{n-1}] \end{bmatrix} \tag{4.8}$$

for the $U(n-1)$ extension of the (9-j) in (4.3) (see also (5.4b)). The relevant operator patterns are inherited from the upper and lower operator patterns, respectively, of the projective operator on the left-hand side. Thus we assert that, with these changes, the results (4.3) and (4.6) are valid in $U(n)$ and completely define two classes of matrix elements for general $U(n)$ tensor operators. Note that these coefficients, unlike U(3), are not necessarily multiplicity-free and, for the results to be meaningful, it is necessary to assume that all multiplicities have been (formally) resolved at the $U(n-1)$ level. Finally, note that (4.3) and (4.6) (when properly generalised to $U(n)$) encompass the set of all elementary unit projective operators for $U(n)$ (Biedenharn and Louck 1968). Some of the corresponding expressions were first computed by Le Blanc and Hecht (1987) using a more restrictive framework.

5. Structural properties of projective operators: limit properties

We have evaluated in § 4 unique and fully labelled projective operators for two classes of $U(n)$ tensor operators. In this construction (§ 3), the operator patterns labelling the projective operators were inherited from their $U(n-1)$ intrinsic substructures. These results are indeed correct, but the labelling for these operators is by no means obvious and requires a formal proof, which we now supply.

The concept of an operator pattern, together with a compatible Gel'fand-Weyl pattern as a unique, canonical labelling for tensor operators has been fully proven in U(3) (Biedenharn *et al* 1967) and is meaningful (though not fully proved as canonical) in $U(n)$. Nonetheless, from a practical point of view, exactly how one determines a given element Γ_{ij} in the operator pattern (Γ) may appear somewhat vague and indirect.

Certainly, no operators have been given, so far, whose eigenvalues are the labels Γ_{ij}, in contrast to the situation for the m_{ij} of the Gel'fand-Weyl pattern (Louck and Biedenharn 1970). We seek to clarify this situation in the present section and remark that the VCS construction aids very much in this effort by supplying matrix elements which will verify the labelling explicitly, as we now show.

For U(2), the labelling is completely determined by the shifts. For U(3) and higher U(n), this is no longer true and we get multiplicities. Nonetheless, the lexical (Γ) patterns precisely enumerate (one-to-one onto) the tensor operators in each multiplicity set. How can one associate a given operator pattern in a multiplicity set to a specific operator with explicit matrix elements? There are two known ways (Louck and Biedenharn 1970).

(a) Operator patterns are associated with the characteristic null space of the operator, i.e. the set of irreps† annihilated by the operator. For example, the generators in U(n) are identified by the fact that the characteristic null space consists *only* of the irrep $[\dot{0}]$.

(b) Operator patterns are determined by their limit properties. Consider the U(n) : U($n-1$) projective operator

$$\begin{bmatrix} (\Gamma)_{n-1} \\ [\boldsymbol{m}_n^{(t)}] \\ (\gamma)_{n-1} \end{bmatrix}$$

labelled by the irrep $[\boldsymbol{m}_n^{(t)}]$ and the two operator patterns $(\Gamma)_{n-1}$ and $(\gamma)_{n-1}$. Using the recursion hypothesis that the U($n-1$) tensorial structure has been resolved, we see that the lower pattern $(\gamma)_{n-1}$ is determined from the expansion (4.1) on the U($n-1$) operator basis. The limit relation can now be given explicitly‡ (Louck and Biedenharn 1973): the U(n) : U($n-1$) projective operator acting on a generic ket vector in U(n) : U($n-1$), i.e.

$$\begin{bmatrix} (\Gamma)_{n-1} \\ [\boldsymbol{m}_n^{(t)}] \\ (\gamma)_{n-1} \end{bmatrix} \left| \begin{matrix} m_{1n} & & \cdots & & m_{nn} \\ & m_{1,n-1} & \cdots & m_{n-1,n-1} & \end{matrix} \right\rangle$$

in the limit $m_{nn} \to -\infty$ obeys the relation

$$\begin{aligned} \lim_{m_{nn}\to-\infty} & \begin{bmatrix} (\Gamma)_{n-1} \\ [\boldsymbol{m}_n^{(t)}] \\ (\gamma)_{n-1} \end{bmatrix} \left| \begin{matrix} m_{1n} & & \cdots & & m_{nn} \\ & m_{1,n-1} & \cdots & m_{n-1,n-1} & \end{matrix} \right\rangle \\ &= \prod_{i=1}^{n-1} (\delta^{\Gamma_{i,n-1}}_{\gamma_{i,n-1}}) \begin{bmatrix} (\Gamma)_{n-1} \\ (\gamma)_{n-1} \end{bmatrix}_{\text{extended}} \left| \begin{matrix} m_{1n} & \cdots & m_{n-1,n} & \\ & m_{1,n-1} & \cdots & m_{n-1,n-1} \end{matrix} \right\rangle . \end{aligned} \tag{5.1}$$

The right-hand side of (5.1) defines a new object in the calculus of tensor operators denoted

$$\begin{bmatrix} (\Gamma) \\ (\gamma) \end{bmatrix}_{\text{extended}} \left| \begin{matrix} m_{1n} & \cdots & m_{n-1,n} & \\ & m_{1,n-1} & \cdots & m_{n-1,n-1} \end{matrix} \right\rangle$$

† The null space qualifier 'characteristic' implies that, if *one* vector in an irrep is annihilated, then *all* vectors of the irrep are annihilated by the operator.

‡ The limit relation in full generality is only implicit in the results of the reference cited; the general result, however, is easily given using the results cited and the product law for projective operators.

on the space of two $U(n-1)$ irreps: $[m_{1n} \ldots m_{n-1,n}]$ and $[m_{1,n-1} \ldots m_{n-1,n-1}]$. We remark that this new function, the limit of projective operator function on the left-hand side of (5.1), inherits as its domain the irrep labels of the factor space $U(n):U(n-1)$, with the single label m_{nn} deleted. We also remark that the product of delta functions on the right-hand side implies that a non-vanishing limit exist only for matrix elements of the unit projective tensor operator for which the condition

$$\Delta w = w^{(f)} - w^{(i)} = 0 \tag{5.2}$$

holds. (This observation could prove crucial for the unambiguous construction of all unit tensors belonging to a given multipicity set. This condition is shown below to hold for the two classes of VCS unit tensors constructed in § 3.)

It is a remarkable result (Louck and Biedenharn 1973, § IV and appendix B) that the limit function exists as a well defined object at the $U(n-1)$ level: the right-hand side of (5.1) is a unitary 6-j operator $U(n-1)$.

Specialising to $U(3)$, the limit in (5.1) yields a unitary Racah function in $U(2)$:

$$\left[\begin{matrix} & & \Gamma_{11} & & \\ M_{13} & \Gamma_{12} & M_{23} & \Gamma_{22} & M_{33} \\ & \gamma_{12} & & \gamma_{22} & \\ & & \gamma_{11} & & \end{matrix}\right] \left| \begin{matrix} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \end{matrix} \right\rangle_{m_{33}\to-\infty}$$

$$= \delta^{\Gamma_{12}}_{\gamma_{12}} \delta^{\Gamma_{22}}_{\gamma_{22}} \left[\begin{matrix} & \Gamma_{11} & \\ \gamma_{12} & & \gamma_{22} \\ & \gamma_{11} & \end{matrix}\right]_{\text{extended}} \left| \begin{matrix} m_{13} & & m_{23} \\ & m_{12} & & m_{22} \end{matrix} \right\rangle \tag{5.3a}$$

$$= \delta^{\Gamma_{12}}_{\gamma_{12}} \delta^{\Gamma_{22}}_{\gamma_{22}} \left\{\begin{matrix} & \Gamma_{11} & \\ \gamma_{12} & & \gamma_{22} \\ & \gamma_{11} & \end{matrix}\right\} \left| \begin{matrix} m_{13} & & m_{23} \\ & m_{12} & & m_{22} \end{matrix} \right\rangle \tag{5.3b}$$

where the notation in (5.3b) has been introduced by Biedenharn and Louck (1981b) in the specific context of an $U(2)$ extended pattern calculus. When acting on the truncated $U(3):U(2)$ factor space (m_{33} deleted), the $U(2)$ invariant extended operator assumes the value

$$\left[\begin{matrix} & \Gamma_{11} & \\ \gamma_{12} & & \gamma_{22} \\ & \gamma_{11} & \end{matrix}\right]_{\text{extended}} \left| \begin{matrix} m_{13} & & m_{23} \\ & m_{12} & & m_{22} \end{matrix} \right\rangle$$

$$= U([0, -w][m_{13}, m_{23}][m_{12}+\Delta_1(\gamma), m_{22}+\Delta_2(\gamma)][\gamma_{12}, \gamma_{22}];$$
$$[m_{12}, m_{22}][m_{13}+\Delta_1(\Gamma), m_{23}+\Delta_2(\Gamma)]) \tag{5.3c}$$

where the parameters (w, $\Delta_i(\gamma)$, $\Delta_i(\Gamma)$) of the unitary (6-j) coefficient have been defined in §§ 2 and 3 (compare with (4.4b)).

It is of interest to remark (Louck and Biedenharn 1973, Le Blanc 1987) that *all* Racah functions in $U(2)$ can be obtained as limits of appropriate $U(3):U(2)$ projection operators. We note the quantity w in (5.3) appears naturally and meaningfully in the VCS framework. This same quantity also appears in the framework of an extended pattern calculus introduced by Biedenharn and Louck but its defining equation (Biedenharn and Louck 1981b, equation (4.45)), although similar to ours, had no direct interpretation there. Since the elementary projective operators are fully known from the pattern calculus (Biedenharn and Louck 1968), one sees that the pattern calculus rules for $U(3)$ allow in a natural way the explicit evaluation of all fundamental (6-j) operators in $U(2)$.

For $U(n)$, $n \geqslant 3$, the set of all (6-j) operators in $U(n)$ can no longer be obtained as limits of $U(n+1):U(n)$ projective operators. One can see this most easily from the structural properties of the (6-j) coefficients in U(n), which involve not only six U(n) irrep labels (associated geometrically with the lines of a tetrahedon) but also four operator patterns (associated with the four triangles in a tetrahedon). The notation (Louck and Biedenharn 1970)

$$\left\langle \begin{matrix} [m_n^{(23)}] \\ (\gamma^{(23)})_{n-1} \end{matrix} \middle| \left\{ \begin{matrix} (\Gamma^{(3)})_{n-1} \\ [m_n^{(3)}] \\ (\gamma^{(3)})_{n-1} \end{matrix} \right\} \middle| \begin{matrix} [m_n^{(2)}] \\ (\gamma^{(2)})_{n-1} \end{matrix} \right\rangle ([m^{(1)}]) \tag{5.4a}$$

for a (6-j) matrix element in U(n) makes these relations evident in that one has explicitly the four operator patterns in evidence. The (implicit) vector coupling relations in (5.4a) are best displayed in the more explicit (9-j) operator form:

$$\begin{bmatrix} (\gamma^{(2)})_{n-1} & & \\ [m_n^{(2)}] & [m_n^{(1)}] & [m_n^{(12)}] \\ [m_n^{(3)}] & [\dot{0}] & [m_n^{(3)}] \\ (\Gamma^{(3)})_{n-1} & & (\gamma^{(3)}) \\ (\gamma^{(23)})_{n-1} & & \\ [m_n^{(23)}] & [m_n^{(1)}] & [m_n] \end{bmatrix} \tag{5.4b}$$

where the operator pattern relevant to a given coupling is identified by the equalities

$$\begin{aligned} &[m_n^{(12)}] = [m_n^{(1)} + \Delta(\gamma^{(2)})] \\ &[m_n^{(23)}] = [m_n^{(2)} + \Delta(\Gamma^{(3)})] \\ &[m_n] = [m_n^{(23)} + \Delta(\gamma^{(23)})] = [m_n^{(12)} + \Delta(\gamma^{(3)})]. \end{aligned} \tag{5.4c}$$

(To remove ambiguity, operator patterns for triangles given by rows (columns) are placed above (below) the irrep.) This (9-j) pattern clearly displays the six irreps and the four operator patterns associated with the four non-trivial vector couplings of a U(n) 'Racah' coefficient.

Comparing this general (6-j) operator with the limit (extended) operator in (5.1) (and with equation (4.8)), one sees that only two of the four operator patterns (($\Gamma^{(3)}$) and ($\gamma^{(3)}$)) occur in (5.1). It has been proven that the two missing operator patterns (($\gamma^{(2)}$) and ($\gamma^{(23)}$)) correspond to multiplicity-free couplings involving, remarkably, a symmetric irrep $[\dot{0}-k]$ (Louck and Biedenharn 1973, (B6b)). VCS theory readily provides a simple explanation for this fact as the Young frame $[m_n^{(1)}]$ is seen to correspond to a symmetric partition $[\dot{0}-w]$. We then have

$$\begin{bmatrix} (\Gamma) \\ (\gamma) \end{bmatrix}_{\text{extended}} \left| \begin{matrix} m_{1n} & \cdots & m_{n-1,n} & \\ & m_{1,n-1} & \cdots & m_{n-1,n-1} \end{matrix} \right\rangle \equiv \begin{bmatrix} [u_{n-1}^{(i)}] & [\dot{0}-w] & [m_{n-1}^{(i)}] \\ [u_{n-1}^{(t)}] & [\dot{0}] & [m_{n-1}^{(t)}] \\ (\Gamma)_{n-2} & & (\gamma)_{n-2} \\ [u_{n-1}^{(f)}] & [\dot{0}-w] & [m_{n-1}^{(f)}] \end{bmatrix}.$$

This also accounts for the fact that the limit relation (5.1) fails to give all (6-j) operators in U($n-1$), except for the special case of U(2) ($n=3$) which is always multiplicity free.

The results obtained in § 4 provide a striking, and simple, exemplification in all U(n) of the limit property given in (5.1). Consider the result in (4.3) specialised to

U(3). The only place the limiting parameter m_{33} occurs is in the K factors. From the explicit form of the K factors, (2.15), we see that in the limit $m_{33} \to -\infty$,

$$\lim_{m_{33}\to-\infty}\left(K\begin{pmatrix}[m_3^{(f)}]\\ [m_2^{(i)}]\end{pmatrix}\left[K\begin{pmatrix}[m_3^{(f)}]\\ [m_2^{(f)}]\end{pmatrix}\right]^{-1}\right)\to\frac{1}{(m_{33})^{\alpha}} \tag{5.5a}$$

with

$$\alpha=\sum_{k=1}^{2}(m_{k3}^{(f)}-m_{k3}^{(i)})-\sum_{k=1}^{2}(m_{k2}^{(f)}-m_{k2}^{(i)})\geqslant 0 \tag{5.5b}$$

implying that the limit is equal to

$$\begin{cases}1 & \sum_{k=1}^{2}(m_{k3}^{(f)}-m_{k3}^{(i)})-\sum_{k=1}^{2}(m_{k2}^{(f)}-m_{k2}^{(i)})=w^{(f)}-w^{(i)}=0\\ 0 & \text{otherwise.}\end{cases} \tag{5.5c}$$

This condition for a non-vanishing limit can be rewritten in the terms of the shift $\Delta_i(\Gamma)=m_{i3}^{(f)}-m_{i3}^{(i)}$ and $\Delta_i(\gamma)=m_{i2}^{(f)}-m_{i2}^{(i)}$, so that we have, for the non-vanishing condition in (5.5c), the relation

$$\sum_{k=1}^{2}\Delta_k(\Gamma)-\sum_{k=1}^{2}\Delta_i(\gamma)=0.$$

Using the definition of the shifts, we see that

$$\sum_{k=1}^{2}\Delta_k(\Gamma)=\Gamma_{12}+\Gamma_{22}=m_{13}^{(t)}+m_{23}^{(t)}$$

(recall $\Gamma_{k2}=m_{k3}^{(t)}$ for the case at hand) and

$$\sum_{k=1}^{2}\Delta_k(\gamma)=\gamma_{12}+\gamma_{22}=m_{12}^{(t)}+m_{22}^{(t)}.$$

Thus the condition for a non-vanishing limit takes the form

$$\sum_{k=1}^{2}(m_{k3}^{(t)}-m_{k2}^{(t)})=0 \tag{5.6a}$$

which—because of the betweeness conditions—implies that each term in (5.6a) is positive and therefore

$$m_{k3}^{(t)}-m_{k2}^{(t)}=0 \qquad k=1,2. \tag{5.6b}$$

This is precisely the delta function condition in (5.1) (recall once more that $\Gamma_{k,n-1}=m_{n3}^{(t)}$ for the case at hand while $\gamma_{k,n-1}=m_{k,n-1}^{(t)}$ always).

A similar limit applied to the maximal $\Delta_n=m_{1n}^{(t)}$ shift VCS operator of (4.6) shows that this result also limits properly to a U($n-1$) (6-j) operator. We thus conclude the VCS evaluation of two classes of projective operators in U(n), (4.3) and (4.6), obey the limit theorem, (5.1), if one assumes that the operator labels have been assigned as written.

Conversely, the desired conclusion that the operator patterns in (3.4) and (3.8) have been correctly assigned follows from the above results. Since the lower pattern in these equations is generic, we may choose the lower pattern in, e.g. (4.3), to be maximal, $m_{k2}^{(t)}=m_{k3}^{(t)}$, and then take the limit $m_{33}\to-\infty$. We find from (5.5) and (5.6) that Γ_{k2} must be $m_{k3}^{(t)}$. (The same considerations apply to (4.6).) This allows us to conclude that the VCS tensor operators in (3.3) and (3.7) at least have a limiting component that carries the assigned operator labels, but it does not establish that the tensors correspond uniquely to the assigned operator patterns. To complete the proof, we need only remark that (4.3) and (4.6) show that the K factors contain the only dependence on m_{nn}. Thus the entire dependence on m_{nn} of (4.3) and (4.6) is contained in a known multiplicative factor. The inverse limit to (5.1) thus exists, and this establishes the desired result. (We remark that more generally the limit $m_{nn} \to -\infty$ yields a $\mathrm{U}(n-1)$ (6-j) operator, for which the subsequent limit $m_{n-1,n-1} \to -\infty$ then yields a $\mathrm{U}(n-1):\mathrm{U}(n-2)$ projective operator, etc. This sequence of limits validates, step by step, the entire operator pattern.) The K factors, which are crucial to this result, are typical of the VCS approach and show once again the significance of the VCS construction.

Finally, we remark that, using the VCS expansion (2.9), it is easily demonstrated that the limit $m_{nn} \to -\infty$ corresponds to the contraction

$$\mathrm{su}(n) \to \mathrm{u}(n-1)[\mathrm{hw}(n-1)] \tag{5.7}$$

of the $\mathrm{su}(n)$ Lie algebra to the semi-direct sum $\mathrm{u}(n-1)[\mathrm{hw}(n-1)]$. The $\mathrm{u}(n-1)$ stability subalgebra, the piecewise sum by components of two $\mathrm{u}(n-1)$ subalgebras, is left intact in the contraction process and this observation readily explains the natural appearance of $\mathrm{U}(n-1)$ recoupling coefficients in the limit.

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